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Bayesian Random Parameter Models of Fertilizer Dosingwith Independent Skew-Normally Distributed Random Components

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Abstract— Random parameter models have been found to better predict the optimum dose of fertilization than fixed parameter models. However, a major restriction of this class of models is that the random parameter components are normallydistributed. This paper introduces random parameter models of fertilizer dosing withindependent skew-normallydistributedrandom componentsusing Bayesian estimation. We compare the Linear Plateau, Spillman-Mitscherlich, and Quadratic random parameter models with random parameter componentswith different distributions, i.e. the Skew-normal, Skew-t, Skew-slash, and Skew-contaminated normal distributions and also their counterparts, i.e., the normal, Student-t, slash and the contaminated normal distributions,with the errors following symmetric normal independent distributions. The method is applied to a dataset of multi-location trials of potassium fertilization of soybeans. The results show that the Student-t Spillman-Mitscherlich Response Model is the best model for soybean yield prediction.

Keywords—Bayesian estimation, Dose-response model, Random parameter model, Skew-normal independent distributions.

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1 Introduction

FERTILIZATION experimental data are typically multisite or multiyear data of different doses of fertilizer applied.A common modeling approach is to fit a general quadratic form to the data by means of least squares under the assumption of a fixed effects model with independent normally distributed random error term with constant variances [1], [2]. However, this approach is unrealistic because it neglectsthe variability that usually exists between sites or years.

An alternative model is the mixed effects approach[3], [4], [5], [6]. This approach allows the parameters to have a random effect component that represent between sites or years variability. The random parameter models have been found to outperform the fixed parameter models to model dose-response relationships[5],[7], [8]. Furthermore, the quadratic functional form commonly used is not always the best model. Tumusiime et al. [7] and [9] showed that the stochastic linear plateau model and the Mitscherlich exponential type functions outperform the quadratic form. In a similar vein, [8] showed that the stochastic linear plateau function is more adequate than the stochastic quadratic plateau function for corn response to Nitrogen fertilizer.

The random parameter components and the errors are usually taken as normally distributedrandom variables [5], [6], [7], [8]. However, the normality and symmetryassumptionsmay be too restrictive because in practice departures from normality is common. Particularly,[10] and [11] concluded that field crop yield distributions are in general non-normal or non-lognormal. The degree of skewness and kurtosis vary by crop type and the amount of nutrients uptake. In addition, (random) weather effects could result in positively or negatively skewed probability functions. Therefore, [12] suggested the beta distribution for the random parameter component of the linear plateau function of wheat response to Nitrogen fertilizer.Furthermore, [13] found that skew normal distribution of the random parameter component outperform the normality assumption.

Lachos,Ghosh, and Arellano-Valle[14] advocated the use of the Skew-normal independent distribution for robust modeling of linear mixed models. The Skew-normal independent distributionisa class of asymmetric, heavy-tailed distributions that includes the Skew-normal distribution,Skew-t, Skew-slash and the Skew-contaminated normal distributions. The class of Skew-normal distributions accomodateobservations with high skewness and heavy tails as well as the normal distribution.

Traditionally, fertilizer-dose response modelsare estimated by means of maximum likelihood estimation (ML)[5], [6], [7], [8]. However, for nonlinear modelsand small sample sizes ML is frequently biased[15]. In addition, convergence can be a problem even with careful scaling and good starting values. Bayesian estimation is an alternative to ML. The advantages of Bayesian estimation arethat the results are valid in small samples and that convergence in the case of nonlinear models is not an issue [12],[15], [16].

The purpose of this paper is Bayesian estimation of random parameter dose (fertilization)-response (yield)

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models for yield data that is Skew-normally independently distributed.

The paper is organized as follows. In Section 2, we introducethe general normal mixed response model,briefly discuss the class of Skew-Normal Independent (SNI) distributions and presentthe SNI-Mixed Model (SNI-MM). Section 3summarizesBayesian inferenceandintroduces model comparison criteria. Section 4 describesthe soybean yield dataset and the response functions. Section 5 presentsthe results and the conclusionsfollow in Section 6.

2 Mixed Effects Model and Skew Normal Independent Distributions

2.1 TheNormal Mixed Effects Model

In general, a Normal mixed effects model reads:

$$Y_i = \eta(\phi_i, X_i) + \epsilon_i, \qquad \phi_i = A_i\beta + B_i b_i, \qquad (1)$$

with

$(b_i, \epsilon_i) \overset{ind}{\sim} N_{n_i+q}\left(0, \mathrm{Diag}(\Sigma, \sigma_e^2 I_{n_i})\right)$,

where the subscript i is the subject index, $i = 1, \ldots, n$; $Y_i = (y_{i1}, \ldots, y_{in_i})^T$ is an$_i$ $\times 1$ vector of $n_i$observed continuous responses for subject $i, \eta_i(\phi_i, X_i) = \{\eta(\phi_i, X_{i1}), \ldots, \eta(\phi_i, X_{in_i})\}^T$ with $\eta(.)$ the nonlinear or linear function of random parameters ϕ_i, and covariate vector X_i, A_i and B_i are known design matrices of dimensions $n_i \times p$ and $n_i \times q$, respectively, β is the $p \times 1$ vector of fixed effects, b_i is the$q \times 1$ vector of random effects, and ϵ_i is the $n_i \times 1$ vector of random errors, andI_{n_i} denotes the identity matrix. The matrices $\Sigma = \Sigma(\alpha)$ with unknown parameter α is the $q \times q$ unstructured dispersion matrix of b_i, σ_e^2 the unknown variance of the error term.When $\eta(.)$ is a nonlinear parameter function, we have the Normal NonLinear Mixed Model (N-NLMM); if $\eta(.)$ is a linear parameter function, we have the N-Linear Mixed Model (N-LMM).

It follows that

$b_i \overset{ind}{\sim} N_q(0, \Sigma)and\epsilon_i \overset{ind}{\sim} N_{n_i}(0, \sigma_\epsilon^2 I_{n_i})$

andthat they are uncorrelated; since $\mathrm{Cov}(b_i, \epsilon_i) = 0$[17], [18].

2.2 Skew-Normal Independent (SNI) Distributions

A skew-normal independent distribution is defined as the p-dimensional random vector $y = \mu + U^{1/2}Z$, where μ is a location vector, Z a multivariate skew-normal random vector with location vector 0, scale matrix Σ and skewness parameter vector λ, i.e. $Z \sim SN_p(0, \Sigma, \lambda)$[14]. Furthermore,U is a positive weight random variable with cumulative distribution function (cdf) $H(u|v)$ and probability density function (pdf) $h(u|v)$, v is a scalar or vector of parameters indexing the distribution of the scale factorU.Given U, Y follows a multivariate skew-normal distribution with location vector 0, scale matrix $u^{-1}\Sigma$ and skewness parameter vector λ, i.e., $Y|U = u \sim SN_p(\mu, u^{-1}\Sigma, \lambda)$.Thus, the SNI distributions are scale mixtures of the skew-normal distributions denoted by $Y \sim SNI_p(\mu, \Sigma, \lambda, H)$.The marginal pdf of Y is

$$f(y) = 2\int_0^\infty \phi_p(y; \mu, u^{-1}\Sigma)\Phi\left(u^{1/2}\lambda^T\Sigma^{-1/2}(y-\mu)\right)dH(u|v),$$

The class of skew-normal independent distributionsis a group of asymmetric heavy-tailed distributionsof robust alternatives to the routinely used of normal distributionsfor mixed effects models[19, 20, 21, 22].A convenient stochastic representation of Y, follows from [20], [21]:

$$Y = \mu + \Delta T + \Gamma^{1/2}T_1, \qquad (2)$$

where$\Delta = \Sigma^{1/2}\delta$, $\Gamma = \Sigma^{1/2}(I - \delta\delta^T)\Sigma^{1/2} = \Sigma - \Delta\Delta^T$, I denotes the identity matrix and $\delta = \lambda/(1+\lambda^T\lambda)^{1/2}, \lambda = \frac{(\Gamma+\Delta\Delta^T)^{-1/2}\Delta}{[1-\Delta^T(\Gamma+\Delta\Delta^T)^{-1}\Delta]^{1/2}}$, $\Sigma = \Gamma + \Delta\Delta^T$, $T = |T_0|, T_0 \sim N_1(0,1)$ and $T_1 \sim N_p(0, I_p)$.

When $\lambda = 0$, the class of SNI distributions reduces to the classof thick-tailed normal independent (NI) distributions[18], [23], [24]. The probability density function(pdf) is $f_0(y) = \int_0^\infty \phi_p(y; \mu, u^{-1}\Sigma)dH(u|v)$,denoted as$Y \sim N_p(\mu, \Sigma, H)$.

2.3. The SNI-Mixed Effects Model

Using the general framework (1), the general SNI-Mixed Model (SNI-MM) is defined as:

$$b_i \overset{iid}{\sim} SNI_q(0, \mathrm{Diag}(\Sigma), \lambda, H) \text{and} \epsilon_i \overset{ind}{\sim} NI_{n_i}(0, \sigma_e^2 I_{n_i}, H),$$
$$i = 1, \ldots, n.$$

where the random effects are assumed to have a multivariate SNI distribution and the random errors are assumed to have a NI distribution.

3 Bayesian Inference

3.1 Prior Distributions and Joint Posterior Density

Below, we apply a Bayesian framework based on the Markov Chain Monte Carlo (MCMC) algorithm to infer posterior parameter estimates.Using (2), thegeneral mixed model can be formulated in hierarchical form for $i = 1, \ldots, n$, as follows:

$Y_i|b_i, U_i = u_i \overset{ind}{\sim} N_{n_i}\left(\eta(A_i\beta + B_i b_i, X_i), u_i^{-1}\sigma_e^2 I_{n_i}\right)$.

$b_i|T_i = t_i, U_i = u_i \overset{ind}{\sim} N_q(\Delta t_i, u_i^{-1}\Gamma)$.

$T_i|U_i = u_i \overset{ind}{\sim} HN_1(0, u_i^{-1})$.

$U_i \overset{iid}{\sim} H(u_i|v)$.

where$HN_1(0, \sigma^2)$ is the half-$N_1(0, \sigma^2)$ distribution, $\Delta = \Sigma^{1/2}\delta$ and $\Gamma = \Sigma - \Delta\Delta^T$, with $\delta = \lambda/(1+\lambda^T\lambda)^{1/2}$ and $\Sigma^{1/2}$ the square root of Σ containing $q(q+1)/2$ distinct elements [18],[20],[25].

Let $Y = (y_1^T, \ldots, y_n^T)^T$, $b = (b_1^T, \ldots, b_n^T)^T$, $u = (u_1, \ldots, u_n)^T$, $t = (t_1, \ldots, t_n)^T$. Then the complete likelihood function associated with $(y^T, b^T, u^T, t^T)^T$, is given by

$$L(\theta|Y, b, u, t) \propto \prod_{i=1}^n [\phi_{n_i}(y_i; \eta(A_i\beta + B_i b_i, X_i), u_i^{-1}\sigma_e^2 I_{n_i})\phi_q(b_i; \Delta t_i, u_i^{-1}\Gamma) \times \phi_1(t_i; 0, u_i^{-1})h(u_i|v)].$$

Tocomplete the Bayesian specification, we need to specify prior distributions for all the unknown parameters $\theta = (\beta^T, \sigma_e^2, \alpha^T, \lambda^T, v^T)^T$. We take$\beta \sim N_p(\beta_0, S_\beta)$, $\sigma_e^2 \sim IG(\tau_e/2, T_e/2)$, $\Gamma \sim IW_q(T_b, \tau_b)$, $\Delta \sim N_p(\Delta_0, S_\Delta)$[18], [20].Forvwe take

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$v \sim Exp(\tau/2)\mathbb{I}_{(2,\infty)}$ for the Skew-t (St) model, $Gamma(a, b)$ for the Skew-slash (SSL) model. Furthermore, $U(0, 1)$ for v_1 and $Beta(\rho_0, \rho_1)$ for v_2 for the Skew-contaminated normal (SCN) model.

Assuming independency for the parameter vector, the joint prior distribution of all unknown parameters is
$\pi(\theta) = \pi(\beta)\pi(\sigma_e^2)\pi(\Gamma)\pi(\Delta)\pi(v)$.

Combining the likelihood function and the prior distribution, the joint posterior density of all unknown parameters is

$$\pi(\beta, \sigma_e^2, \Gamma, \Delta, b, u, t|y) \propto \prod_{i=1}^{n}[\phi_{n_i}(y_i; \eta(A_i\beta + B_i b_i, X_i), u_i^{-1}\sigma_e^2 I_{n_i})\phi_q(b_i; \Delta t_i, u_i^{-1}\Gamma) \times \phi_1(t_i; 0, u_i^{-1})h(u_i|v)]\pi(\theta).$$

3.2 Model Comparison Criteria

The expected Akaike information criterion (EAIC) and the expected Bayesian information criterion (EBIC) are a deviance-based measure appropriate for Bayesian model selection[26], [27].Let θ and $Y = (y_1, \dots, y_n)^T$ be the entire model parameters and data, respectively. Define $D(\theta) = -2\ln f(y|\theta) = -2\sum_{i=1}^{N} \ln f(y_i|\theta)$, where $f(y_i|\theta)$ is the marginal distribution of y_i.Then $E[D(\theta)]$ is a measure of fit and can be approximated by using the MCMC output in a Monte Carlo simulation. This measure isobtained as given by $\overline{D} = \frac{1}{K}\sum_{k=1}^{K} D(\theta^{(k)})$. Where $\theta^{(k)}$is the k^{th} iteration of MCMC chain ofthe model and K is the number ofiterations.

The EAIC andEBIC define as follows
$\widehat{EAIC} = \overline{D} + 2p$ and $\widehat{EBIC} = \overline{D} + p\log(N)$,
where $\overline{D}$ is the posterior mean of the deviance, p is the number of parameters in the model, N is the total number of observations.These criteria penalizing models with more complexity.Smaller value of EAIC and EBIC indicate a better fit[20].

4 Case Study

4.1 Data

The datasetis obtained from 19 multi-location trials of potassium fertilization of soybeans. The experiments were carried out between 2002 and 2014. The soil types are Ultisols, Inceptisols, Vertisols, and Oxisols with soil potassium contents varying from very low to very high. Common soybean varieties were used.Each experimentsconsisted of five levels of potassium fertilization. The doses applied were 0, 40, 80, 160 and 320 kg ha-1 of KCl .The plots were 6 by 5 m, or 4 by 5 marranged in a randomized complete block design with three to nine replications. The response variable was soybean yield (t ha-1). The yields reported are averagesover replications[28], [29], [30], [31].

4.2 Response Functions

We consider three response functions: the Linear Plateau (LP), the Spillman-Mitscherlich (SM) and the Quadratic functions (Q).

The stochastic LPisdefined as follows:

$$Y_i = \min(\alpha_1 + (\alpha_2 + b_{2i})X_i;\ \mu_p + b_{3i}) + b_{1i} + \varepsilon_i \quad (3)$$

where forlocationi, Y_i is the soybean yield;X_i the potassium fertilizer dose; α_1 the intercept parameter; α_2 the linear response coefficient; u_p the plateau yield; b_{1i}, b_{2i},and b_{3i} are the random effects; and ε_i is the random error term. In term of (1),= $(\alpha_1, \alpha_2, \alpha_3)^T b_i = (b_{1i}, b_{2i}, b_{3i})^T$; $b_i \overset{iid}{\sim} SNI_q(0, \Sigma, \lambda, H)$ and $\epsilon_i \overset{ind}{\sim} NI_{n_i}(0, \sigma_e^2 I_{n_i}, H)$.

The stochastic SM reads:

$$Y_i = \beta_1 - (\beta_2 + b_{2i})\exp((-\beta_3 + b_{3i})X_i) + b_{1i} + \varepsilon_i \quad (4)$$

where β_1 is the maximum yield attainable by potassium fertilization; β_2 is the yield increase; β_3 is the ratio of consecutive incrementsof the yield; all other parameters, variables and distributions as in (3).

The stochastic Q isdefined as:

$$Y_i = \gamma_1 + (\gamma_2 + b_{2i})X_i + (\gamma_3 + b_{3i})X_i^2 + b_{1i} + \varepsilon_i \quad (5)$$

where γ_1 is the intercept parameter whose position (value) can be shifted up or down by the random effect b_{1i}; γ_2 is the linear response coefficient with random effect parameter b_{2i}; γ_3 is the quadratic response coefficient whose position can be shifted up or down by the random effectb_{3i};$\gamma = (\gamma_1,$ γ2, γ3T;all other variables and distributions as in (3)[7], [8], [9].

4.3 Statistical Analysis

The dataset was used to identify the model with the best fit among the random parameter models of fertilizer dosing. The models differed with respect to the distributionsof the random effects and random errors. Specifically:

Model 1: Skew-normal distribution for the random effects and Normal distribution for the random errors (SN-N)
Model 2: Skew-t distribution for the random effects and t distribution for the random errors (St-t)
Model 3: Skew-slash distribution for the random effects and slash distribution for the random errors (SSL-SL)
Model 4: Skew-contaminated normal distribution for the random effects and contaminated normal distribution for the random errors (SCN-CN).
Model 5: Normal distribution for the random effects and random errors (N-N)
Model 6: t distribution for the random effects and random errors (t-t)
Model 7: Slash distribution for the random effects and slash distribution for the random errors (SL-SL)
Model 8: Contaminated normal distribution for the random effects and contaminated normal distribution for the random errors (CN-CN).

The following independent priors were considered to perform the Gibbs sampler, $\alpha_k \sim N(0, 10^3)$,$\beta_k \sim N(0,$ 103,γk ~N0, 103, σ2~ IG0.1, 0.1,Γ~ IG0.1, 0.1,Δ~ N0, 0.001, and $v \sim Exp(0.1)I(2,)$for the skew-t and t-model; $v \sim Gamma(0.1,0.01)$for the skew-slash model and slash model;$\nu_1 \sim Beta(1, 1)$ and $\nu_2 \sim Beta(2, 2)$ for the skew-contaminated normal and contaminated normal model.

For each of the models, we ran three parallel independent

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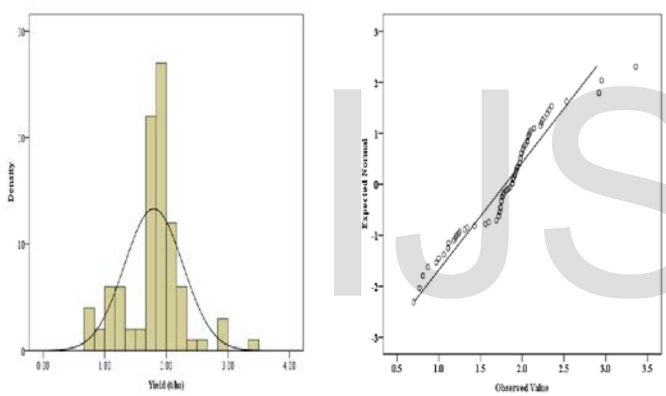
chains of the Gibbs sampler of50 000 iterations for each parameter with thinning of 5 and initial burn in of 25 000. However, for the SSL-SL and SCN-CN model, convergence was achieved at 75 000 iterations. We monitored chain convergence using trace plots, autocorrelation plots and the Brooks-Gelman-Rubin scale reduction factor ($\hat{R}$)[32]. To avoid non-convergence, we normalized the original doses (subtracted the mean and divided by the standard deviation) [33] which gave: -1.06, -0.70, -0.35, 0.35, and 1.76, respectively. We fitted the models using the R2jags package available in R [34].

5 Results and Discussion

5.1 SoybeanYield Data

Fig. 1 shows the histogram and normal Q-Q plot of soybean yield data while the boxplot is presented in Fig. 2.The figures indicate that soybean yield is non-normally distributed. It is skewed with heavy-tails. In particular, the Q-Q plot does not followa straight line,whilethe boxplot showsasymmetryand outliers.Thus, it seems appropriate to fit a skewed heavy-tailed model to the data.



(a) (b)

Fig. 1. Soybean yield data: (a) Histogram; (b) Normal Q-Q plot.

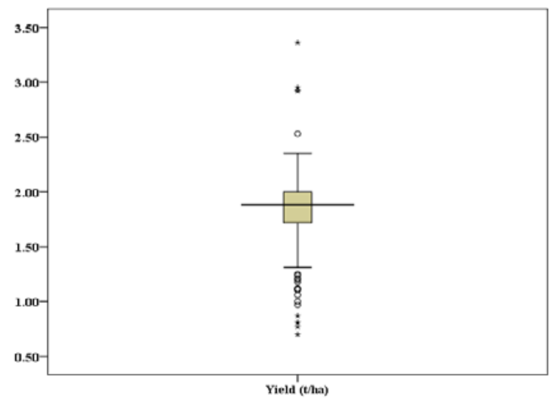


Fig. 2.Boxplot of soybean yield data.

5.2Linear Plateau Response Models

Based on the EAIC and the EBIC in Table 1, we find that among the SNI modelsthe Skew-t(St-t) Model gives the best fit, followed by the Skew-slash (SSL-SL), Skew-contaminated normal (SCN-CN) and Skew-normal (SN-N) Model.Among the NI models, the Student-t (t-t) Model gives the best fit, followed by the contaminated normal (CN-CN), slash (SL-SL), and normal (N-N) Model. We furthermore find that the heavy-tailed distributions outperform the skew heavy-tailed ditributionsand the normal distributions, except for the Skew-t distribution. However, the t-t Model outperforms the St-tModel. Moreover, the asymmetry parameters ($\lambda_1, \lambda_2, \lambda_3$) of Skew-t model are not significant. Thus, the t-tModelis the best Linear Plateau Response Model.

TABLE 1.THE LINEAR PLATEAU MODELS

Parameter	SN-N		S*t-t*		SSL-SL		SCN-CN	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
α_1	1.471	0.113	1.533	0.143	1.469	0.151	1.509	0.159
α_2	29.193	19.320	29.417	19.604	29.657	19.531	28.982	18.929
μ_p	1.880	0.134	1.843	0.217	1.826	0.234	1.882	0.246
σ^2_ε	0.020	0.004	0.015	0.004	0.013	0.004	0.014	0.005
d_1	0.250	0.114	0.173	0.089	0.148	0.076	0.160	0.093
d_2	49.074	237.868	30.677	129.636	30.903	121.481	26.718	92.546
d_3	0.121	0.074	0.071	0.045	0.062	0.039	0.069	0.046
λ_1	0.056	0.467	-0.006	0.262	-0.024	0.265	0.017	0.288
λ_2	0.059	0.654	0.005	0.338	-0.015	0.325	0.003	0.367
λ_3	0.175	1.036	-0.007	0.516	-0.039	0.529	0.036	0.573
ν (ν_1)			5.834	3.018	2.846	1.748	0.473	0.265
ν_2							0.513	0.215
EAIC	-85.78		**-98.03**		-89.47		-88.35	
EBIC	-86.00		**-98.28**		-89.71		-88.61	
Parameter	N-N		*t-t*		SL-SL		CN-CN	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD

α_1	1.473	0.114	1.555	0.109	1.482	0.110	1.521	0.125
α_2	39.968	20.482	29.274	19.480	30.780	19.899	29.061	19.030
μ_p	1.878	0.129	1.853	0.111	1.854	0.122	1.867	0.119
σ^2_ε	0.139	0.014	0.014	0.004	0.012	0.004	0.012	0.006
d_1	0.467	0.095	0.374	0.098	0.355	0.085	0.340	0.113
d_2	13.135	11.964	2.534	4.694	3.684	6.048	2.452	3.937
d_3	0.306	0.081	0.241	0.074	0.227	0.068	0.227	0.077
ν (ν_1)			5.043	3.039	2.697	1.620	0.515	0.253
ν_2							0.450	0.226
EAIC	-93.56		**-104.91**		-95.86		-97.21	
EBIC	-93.71		**-105.09**		-96.04		-97.41	

Table 1furthermore shows that for the t-tModel,all the fixed effects, i.e., the intercept parameter (α_1), the linear response coefficient (α_2), the plateau yield $u_p$and the random effects(d_1, d_2, d_3) are significant.

5.3Spillman-Mitscherlich Response Models

Based on the EAIC and EBIC in Table 2 we find the following rankings of the SNI and NI models: St-t <SSL-SL <SN-N <SCN-CN and t-t <N-N <SL-SL <CN-CN. As in Section 5.2.we observe that the heavy-tailed distributionsoutperform the skew heavy-tailed distributions and that the asymmetry parameters ($\lambda_1, \lambda_2, \lambda_3$) of St-tModel are not significant.It furthermore follows that the t-t Modelis the best Spillman-Mitscherlich Response Model.

TABLE 2.THE SPILLMAN-MITSCHERLICH MODELS

Parameter	SN-N		S*t-t*		SSL-SL		SCN-CN	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
β_1	1.982	0.110	1.924	0.095	1.949	0.106	1.960	0.109
β_2	0.073	0.025	0.059	0.033	0.074	0.036	0.073	0.033
β_3	1.762	0.294	1.771	0.311	1.699	0.310	1.740	0.313
σ^2_ε	0.013	0.003	0.010	0.003	0.009	0.003	0.011	0.003
d_1	0.220	0.085	0.148	0.072	0.133	0.064	0.170	0.077
d_2	0.005	0.005	0.004	0.002	0.004	0.002	0.004	0.002
d_3	0.364	0.289	0.283	0.287	0.275	0.240	0.315	0.269
λ_1	0.008	0.101	-0.002	0.071	0.000	0.069	0.001	0.063
λ_2	0.061	0.785	-0.013	0.441	0.003	0.423	0.004	0.434
λ_3	0.006	0.105	-0.001	0.086	0.000	0.067	0.001	0.067
ν (ν_1)			5.413	2.733	3.197	1.769	0.317	0.235
ν_2							0.549	0.205
EAIC	-128.10		**-144.88**		-130.58		-127.70	
EBIC	-128.33		**-145.13**		-130.83		-127.96	
Parameter	N-N		*t-t*		SL-SL		CN-CN	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
β_1	1.950	0.111	1.919	0.092	1.945	0.103	1.955	0.106
β_2	0.032	0.013	0.054	0.021	0.069	0.024	0.067	0.023
β_3	2.495	0.415	1.848	0.312	1.765	0.300	1.808	0.305
σ^2_ε	0.104	0.010	0.009	0.003	0.009	0.003	0.010	0.003
d_1	0.465	0.087	0.371	0.087	0.355	0.080	0.401	0.086
d_2	0.008	0.006	0.053	0.012	0.051	0.012	0.052	0.012
d_3	0.623	0.199	0.441	0.240	0.463	0.198	0.490	0.222
ν (ν_1)			4.754	2.731	3.139	1.735	0.321	0.235
ν_2							0.544	0.203
EAIC	-147.72		**-153.88**		-138.45		-135.19	
EBIC	-147.88		**-154.06**		-138.63		-135.39	

For the t-t Model, the fixed effects, i.e., the maximum yield coefficient (β_1), the increase in yield (β_2), the ratio of successive increment (β_3)and the random effects(d_1, d_2, d_3) are significant.

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5.4TheQuadratic Response Models

Comparison of the EAIC and EBICin Table 3 leads to the following rankings: St-t < SSL-SL < SCN-CN < SN-N andt-t < SL-SL <CN-CN <N-N.The results furthermore show that the heavy-tailed distributionsoutperform the skew normal and normal distribution, that the heavy-tailed distributionsare better than the normal ones and that overallthe t-t Modelisthe best Quadratic Response Model.

TABLE 3.THE QUADRATIC MODELS

Parameter	SN-N		S*t*-*t*		SSL-SL		SCN-CN	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
γ_1	1.794	0.126	1.810	0.088	1.786	0.102	1.788	0.101
γ_2	0.506	0.101	0.351	0.063	0.401	0.080	0.393	0.081
γ_3	-0.389	0.101	-0.261	0.059	-0.298	0.075	-0.293	0.074
σ^2_ε	0.031	0.021	0.016	0.005	0.012	0.005	0.014	0.006
d_1	0.203	0.233	0.130	0.066	0.096	0.050	0.110	0.064
d_2	0.021	0.227	0.008	0.004	0.007	0.004	0.007	0.004
d_3	0.018	0.232	0.006	0.005	0.005	0.003	0.006	0.003
λ_1	0.014	0.288	0.000	0.064	0.000	0.065	0.000	0.068
λ_2	0.026	0.699	0.000	0.267	-0.001	0.249	0.000	0.257
λ_3	0.021	0.666	0.000	0.300	-0.001	0.277	-0.001	0.286
ν (ν_1)			3.764	1.335	1.682	0.956	0.365	0.165
ν_2							0.275	0.152
EAIC	-42.49		**-91.03**		-76.20		-73.51	
EBIC	-42.71		**-91.27**		-76.45		-73.78	
Parameter	N-N		*t*-*t*		SL-SL		CN-CN	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
γ_1	1.796	0.107	1.825	0.079	1.783	0.096	1.788	0.096
γ_2	0.510	0.072	0.330	0.056	0.398	0.072	0.391	0.076
γ_3	-0.386	0.072	-0.246	0.052	-0.297	0.067	-0.292	0.069
σ^2_ε	0.033	0.006	0.014	0.005	0.011	0.005	0.014	0.006
d_1	0.445	0.085	0.327	0.085	0.297	0.074	0.315	0.092
d_2	0.046	0.030	0.082	0.023	0.075	0.020	0.077	0.021
d_3	0.030	0.022	0.072	0.019	0.066	0.017	0.068	0.018
ν (ν_1)			2.516	1.329	1.617	0.868	0.366	0.160
ν_2							0.265	0.141
EAIC	-45.32		**-104.44**		-83.22		-80.59	
EBIC	-45.47		**-104.62**		-83.40		-80.79	

For the t-tModel,all the fixed effects, i.e., the intercept parameter (γ_1), the linear response coefficient (γ_2), the quadratic response coefficient (γ_3), and the variance component (d_1, d_2, d_3) are significant.

5.5Comparingthe Linear Plateau, Spillman-Mitscherlich and Quadratic Models

Comparing the Linear Plateau (LP), Spillman-Mitscherlich(SM) and Quadratic (Q) models under eight distributional assumptions, we find that the t-tSpillman-Mitscherlich model has the smallest EAIC and EBIC values among the competing models indicating that this is the best fit model for the soybean yield data (Table 4). The correlation between observed and fitted values is significant at the 0.01 level (r=0.984**) (Fig. 3).

TABLE 4.COMPARISON OF LP, SM AND Q MODELS

Distribution	LP		SM		Q	
	EAIC	EBIC	EAIC	EBIC	EAIC	EBIC
SN	-85.78	-86.00	-128.10	-128.33	-42.49	-42.71
S*t*	**-98.03**	**-98.28**	**-144.88**	**-145.13**	**-91.03**	**-91.27**
SSL	-89.47	-89.71	-130.58	-130.83	-76.20	-76.45
SCN	-88.35	-88.61	-127.70	-127.96	-73.51	-73.78
N	-93.56	-93.71	-147.72	-147.88	-45.32	-45.47
Student-*t*	**-104.91**	**-105.09**	**-153.88**	**-154.06**	**-104.44**	**-104.62**

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SL	-95.86	-96.04	-138.45	-138.63	-83.22	-83.40
CN	-97.21	-97.41	-135.19	-135.39	-80.59	-80.79

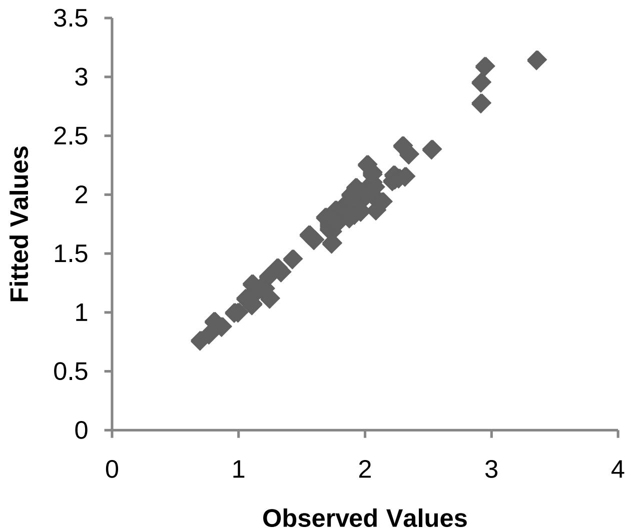

Fig. 3.Fitted values vs observed values plot of Student-*t* SM model

6CONCLUSION

We investigated the performance of linear and nonlinear mixed response modelswith Skew normal independent (SNI) and normal (NI) distributions of random effects. We applied the Bayesian estimation framework to a dataset of multi-location trials of potassium fertilization of soybeans. We compared the Linear Plateau, Spillman-Mitscherlich, and Quadratic random parameter models with different distributions of the random parameter component, i.e. the Skew-normal, Skew-*t*, Skew-slash, and Skew-contaminated normal distributions and also theirsymmetric counterparts, i.e., the normal, Student-*t*, slash and the contaminated normal distributions with the errors following symmetric normal independent distributions.

The overall results showed that for all three models of fertilizer dosing, the heavy-tailed distributions and skew heavy-tailed distributionsoutperform the normal and skew normal distributions. Furthermore, the heavy-tailed distribution approach performed better than the skewed version. The best model for soybean yield predictionturned out to be the Student-*t*Spillman-Mitscherlich Response Model.

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